Line geometry and 3D graphic statics

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Preliminary note

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The mathematical basis for the line geometry, with the line as its basic element, is presented in the paper because the forces that act on a body in space - whether we are interested in conditions of movement or immobility (balance) of that body - are vectorial values related to specific lines of action, so that the system of forces can be linked to the set of lines in space. The application of line geometry has proven the known claim that the system of forces in space, provided the forces are in a general position, can be reduced to two forces on reciprocal lines. The visualisation of this claim via the 3D graphic statics is also presented.

Key words:

extended Euclidean space, polarity, null system, line system, Plücker coordinates, linear complex

Prethodno priopćenje

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Pravčasta geometrija i prostorna grafostatika

Budući da su sile koje djeluju na tijelo u prostoru, bilo da nas zanimaju uvjeti kretanja ili mirovanja (ravnoteže) toga tijela, vektorske veličine vezane za određene pravce djelovanja, tako da se sustav sila može povezati sa skupom pravaca u prostoru, u radu je ukratko prikazana matematička osnova pravčaste geometrije koja kao osnovni element uzima pravac. Primjenom pravčaste geometrije dokazana je poznata tvrdnja da se sustav sila u prostoru, ako su one u općemu položaju, može svesti na dvije sile na recipročnim pravcima. Prikazana je i vizualizacija te tvrdnje metodom prostorne grafostatike.

Ključne riječi:

prošireni euklidski prostor, polaritet, ništični sustav, pravčasti prostor, Plückerove koordinate, linearni kompleks

Vorherige Mitteilung

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Projektive Geometrie und räumliche Graphostatik

Da die Kräfte, die auf einen Körper im Raum einwirken, unabhängig davon, ob wir an den Bedingungen der Bewegung oder der Ruhe (Gleichgewicht) dieses Körpers interessiert sind, Vektorgrößen sind, die auf bestimmte Wirkrichtungen bezogen sind, so dass das Kräftesystem auf eine Reihe von Richtungen im Raum bezogen werden kann, wird in dieser Arbeit in kurzen Zügen die mathematische Basis der projektiven Geometrie dargestellt, die als Grundelement die Richtung annimmt. Es ist allgemein bekannt, dass das Kräftesystem im Raum, wenn die Kräfte sich in einer allgemeinen Position befinden, unter Anwendung der projektiven Geometrie auf zwei Kräfte in wechselseitigen Richtungen reduziert werden kann. Die Visualisierung dieser Behauptung durch die Methode der räumlichen Graphostatik wird ebenfalls dargestellt.

Schlüsselwörter:

erweiterter euklidischer Raum, Polarität, Nullsystem, projektiver Raum, Plücker-Koordinaten, linearer Komplex

1. Introduction

Line geometry is a type of geometry where line is a basic element and its main objects of study are line formations such as line complexes, congruences and ruled surfaces, all containing infinitely many lines. Line geometry emerged from the study of statics and kinematics of rigid bodies, since forces acting along different parallel lines have different effects and the rotations about parallel but different axes bring the body into different positions.

A. F. Möbius, in his paper [1] from 1833, showed that a general system of forces in space can be replaced by two forces acting on conjugate lines of a null polarity. Null polarity is a type of polarity, a transformation of the projective plane which was first defined via conics using synthetic method in 16th century by G. Desargues, and polarities in the projective space defined using quadrics were studied in the 18th century by G. Monge and his student J. V. Poncelet, who is considered to be the founder of projective geometry, in the 19th century.

Möbius had an analytical geometrical approach, but all the terminology that is used to describe his results, which is also commonly used in this field of research still today, comes from J. Plücker (Figure 1, right) and his paper [3] from 1865. This paper is the conception of line geometry.

Nevertheless, such novelty could not come to life without the work of H. Grassmann (Figure 1, left), who first showed that other figures in space, beside points, which were until then only defined as loci of points given by an equation, can be given coordinates and we shall see this applied to lines. His book [4] from 1848 is considered to be the foundation of geometrical (and linear) algebra. Independently, definition of a line with six coordinates was introduced by A. Cayley in his paper [5] from 1860.
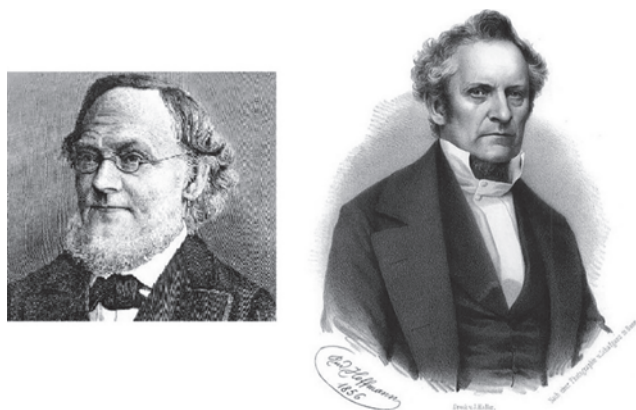

Figure 1. Hermann Grassmann (1809. - 1877.) (left) and Julius Plücker (1801. - 1868.)(right) [6, 7]

Plücker was a mathematician but also a physicist, as were most of the mathematicians in those days; his theory of line space was inspired by mechanics, as can be seen in his paper [8] from 1866 where he connects system of forces to line complexes. We must mention F. Klein, one of the greatest mathematicians

of 19th century, and his commentary [9] on Plücker's papers from 1871, where he clarified certain perplexities and further connected line space geometry with rigid body mechanics.

Although mechanics was the starting point for line geometry, development of the geometry brought it further from mechanics and its results were no longer applied to practical problems. At the same time, unrelated to line geometry, graphic statics developed through papers by W. J. M. Rankinea [10], J. C. Maxwella [11, 12], C. Culmanna [13] and L. Cremone [14], after it first appeared at the end of 16th century in the work of S. Stevin (parallelogram and triangle of forces, the concept of funicular polygon) and P. Verignon (funicular polygon) in the 17th century. Maxwell and Cremona proved the validity of the construction of reciprocal figures (form diagram/funicular polygon and force diagram) by treating these plane figures as projections of dual polyhedra (such that vertices of one lie on the sides of the other and vice versa, (pair of dual tetrahedrons was discovered by Möbius [1]) but otherwise, all the procedures of graphical statics were planar. Regardless of this limitation, graphic statics was a very popular tool for engineers in the last third of 19th century and the first third of 20th century.

There are three possible reason why graphic statics then, except in rare occasions [15, 16], wasn't developed for space. Graphical procedures in space are much more complex, since they have to be done using (at least two) plane projections of spatial systems and visualizations of real spatial relations ("return" from drawing to space) can be very difficult, for some even impossible, task. Furthermore, if we exclude the problem of finding equilibrium, a system of forces in plane can be replaced either by one resultant force or by one resulting moment (force couple) whereas in space there is a third possibility: resulting force and resulting moment (with respect to a chosen point); only if the resulting moment is orthogonal to the resulting force can the system be replaced by a resultant.

Development of computers, computer graphics tools and 3D computer modelling brought forward the revival of graphic statics and the possibility for its application in spatial situations. Many papers were written on the extension of graphic statics to space and two main approaches emerged, polyhedral and vector. In the polyhedral approach, each force is represented by the side of a polyhedron orthogonal to the force's line of action such that the intensity of the force equals the area of the side, [17, 18]. The vector approach is the direct extension of planar graphic statics to space: the forces are represented with vectors [19]. This way, the main property of graphic statics is preserved – perception of spatial relations. Namely, the aim of the "new" computer graphic statics is, for the most part, to create free-form but equilibrating constructions [20]. The theoretical basics for construction of reciprocal diagrams, which rely heavily on projective geometry, are given in [21].

Line geometry provides the third possible approach to spatial graphic statics. As was shown in papers by Möbius, Plücker and Cayley, line geometry gives the theoretical background to the constructive geometric procedures for dealing with forces and

on the other hand, with the use of Grassmann algebra, these procedures can be translated to algebraic expressions and then into program code.

Geometry of line space is presented in the third section, and the basics of projective geometry, necessary for its introduction, are briefly presented in the second section. As an example of application of line geometry in statics, we prove the known claim that a general system of forces in space can be replaced by two forces in the fourth section using line geometry based argument and show the solution to this problem via graphic statics.

2. Projective extension of Euclidean space

From the beginning of our education, we deal with Euclidean geometry – we learn the basic planar constructions using ruler and compass and later solve more complex planar and spatial problems using analytical method. We study such notions as distance, angle, parallelism and orthogonality. But there exist a variety of different types of geometries, and the one we will introduce and use in this paper doesn't care about these notions and the only tool necessary for constructions is the ruler.

Projective geometry had a gradual development starting from the 15th century, being the central topic for many mathematicians. Motivation for this type of geometry comes from renaissance fine arts, from geometric rules of central projection (perspective drawing) which is incorporated in the style of many renaissance paintings. In Da Vinci's masterpiece *The Last Supper* (Figure 2) we can notice the main characteristic of this projection – parallel lines in space meet at a finite point in the drawing.
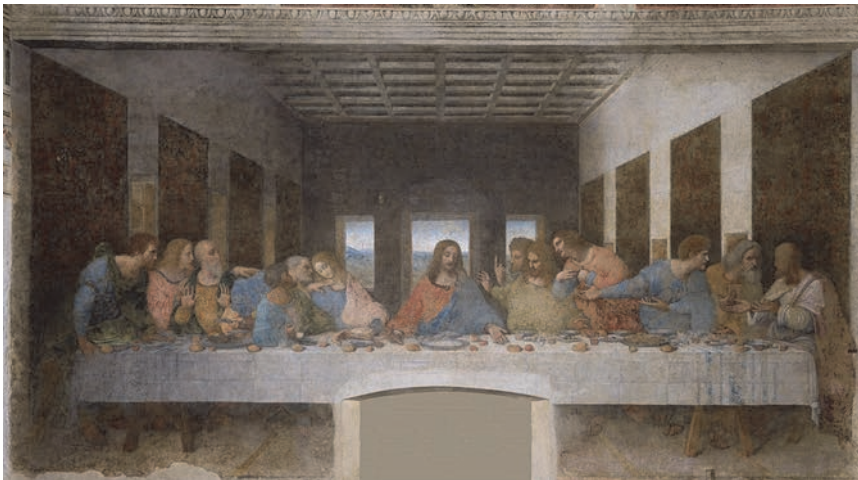

Figure 2. Leonardo da Vinci: *The Last Supper*, (1495-1498) [23]

2.1. Projective space

Historically, and still in the literature today, projective plane is studied in great detail and then the theory is extended to *n* dimensions [22, 24]. In this paper we deal exclusively with three-dimensional projective space. Therefore, we will omit the adjective "three-dimensional".

Basic elements of projective space are same as in the Euclidean space, namely, points, lines and planes. On the set of all basic elements we define the *incidence relation*, relation of belonging. For instance, the point A is incident to the line p if the point A belongs, lies on the line p, and symmetrically, the line p is incident with the point A if the line p passes through the point A. Beside the basic elements and the incidence relation, certain axioms are given to secure we have "enough" basic elements and to formally describe the incidence relation [24]. Such structure is called *projective space.* As an example, axiom A6 [24] guarantees the existence of elements outside one plane:

Axiom A6: *If* A, B *and* C *are three non-collinear points (not all three incident to the same line), then there must exist a point* D *outside the plane determined by* A, B *and* C *(not all four incident to the same plane).*

The most important notion of projective geometry is *duality*, discovered simultaneously by J. V. Poncelet and J. D. Gergonne in the 19th century. Its statement in the projective space is as follows.

Principle of duality in projective space: *In every true statement of the projective geometry of space we can replace the notion of* point *with the dual notion of* plane *and vice versa, while the notions of* line *and* incidence *remain the same. The new statement will remain a true statement of projective geometry of space. Such two statements are called* dual statements.

As an example, the statement dual to axiom A6 states: *If α, β and γ are three planes that do not contain a same line (not all three incident to the same line), then at least one plane δ exists that does not contain the common point of intersection of planes α, β and γ (not all four incident to the same point).*

2.2. Extended Euclidean space

When we have at hand an abstract structure, as we currently have the projective space defined with axioms, we desire to find a geometric realization, a *model* of this structure.

From the beginning of projective geometry, mathematicians wanted to find a model of projective space connected to the usual Euclidean space. We will now present the historical construction of extension of Euclidean space into projective space [24], which is still in use today, especially in the field of descriptive geometry [25]. We denote by $\mathbf{E}^3$ the real Euclidean space.

To every line of $\mathbf{E}^3$ we add, extend it by, one *ideal point* (point at infinity). For every line, its ideal point is its point of intersection with every line parallel to it.

To every plane of $\mathbf{E}^3$ we add, extend it by, one *ideal line* (line at infinity). This ideal line of a plane contains all ideal points of all

lines in that plane and it is the intersection line of that plane with all planes parallel to it.

Finally, to the space $\mathbf{E}^3$ we add, extend it by, one *ideal plane* (plane at infinity). This ideal plane contains all ideal lines of all planes in the space $\mathbf{E}^3$ and all ideal points of all lines in the space $\mathbf{E}^3$.

It can be proven that this extension satisfies all axioms of the projective plane; this model is called *the extended Euclidean space* and we will denote it by $\mathbf{P}^3(\mathbb{R})$. In $\mathbf{P}^3(\mathbb{R}))$ there is no distinction between ideal and non-ideal elements.

2.3. Analytical model; homogenous coordinates

The axiomatic approach is limited to ruler constructions and for more complex needs we require more powerful tools. For this reason we introduce coordinates in $\mathbf{P}^3(\mathbb{R})$ [24].

The standard Cartesian coordinate system in $\mathbf{E}^3$ consists of the origin point $O(0, 0, 0)$ and three pair-wise orthogonal coordinate axes (abscissa or axis x_1, ordinate or axis x_2 and applicate or axis x_3) with prescribed unit lengths. Every point in $\mathbf{E}^3$ is represented with three coordinates (x_1, x_2, x_3) which are precisely its projections to the coordinate axes. These coordinates are called *affine coordinates of a point*.

Points in the extended Euclidean space $\mathbf{P}^3(\mathbb{R})$ have, beside the three mentioned, another coordinate, usually placed in the first place, so that the point in $\mathbf{P}^3(\mathbb{R})$ is represented with 4 coordinates $x_0, x_1, x_2, x_3,$ such that not all $x_1 \in R$ are 0 and:

$$\lambda (x_0{:}x_1{:}x_2{:}x_3) = (\lambda x_0{:}\lambda x_1{:}\lambda x_2{:}\lambda x_3) = (x_0{:}x_1{:}x_2{:}x_3), \lambda \neq 0, \lambda \in \mathbb{R} \quad (1)$$

Thus, the point is represented by a class of ordered four-tuples of real numbers $(x_0{:}x_1{:}x_2{:}x_3)$ that are called *homogenous coordinates of a point*.

Ideal points are characterized by $x_0 = 0$ and using (1) we write their coordinates in the form $\left(0{:}1{:}\frac{x_2}{x_1}{:}\frac{x_3}{x_1}\right)$, which shows they must lie in one plane since they make a two-parameter set of points. Non-ideal points must have $x_o \neq 0$ and it is common to put $x_0 = 1$. They are in bijective correspondence with points of $\mathbf{E}^3$, from the homogenous coordinates $(1{:}x_1{:}x_2{:}x_3)$ of a point in $\mathbf{P}^3(\mathbb{R})$ we can read out the affine coordinates (x_1,x_2,x_3) of the same point observed as a point in space $\mathbf{E}^3$.

In the analytic, point based geometry of $\mathbf{E}^3$, the plane is given as the loci of points given by the equation:

$$Ax_1 + Bx_2 + Cx_3 = D \quad (2)$$

From this equation we read that a plane intersects the coordinate axes at points with affine coordinates $\left(\frac{D}{A},0,0\right)$, $\left(0,\frac{D}{B},0\right)$ and $\left(0,0,\frac{D}{C}\right)$, and following the definition of point coordinates we may try to use the triplet $\left(\frac{D}{A},\frac{D}{B},\frac{D}{C}\right)$ as plane coordinates until we realize that all planes through the origin point would be represented by the same triplet $(0, 0, 0)$.

On the other hand, in the extended Euclidean space $\mathbf{P}^3(\mathbb{R})$ a plane is dual to a point and we can introduce coordinates for planes in the following way; every plane is represented as a class of ordered four-tuples of real numbers $(v_0{:}v_1{:}v_2{:}v_3)$ such that not all $v_i \in \mathbb{R}$ are 0 and:

$$\lambda (v_0{:} v_1{:} v_2{:} v_3) = (\lambda v_0{:} \lambda v_1{:} \lambda v_2{:} \lambda v_3) = (v_0{:} v_1{:} v_2{:} v_3), \lambda \neq 0, \lambda \in \mathbb{R} \quad (3)$$

A point $\mathbf{x} = (x_0{:}x_1{:}x_2{:}x_3)$ and a plane $v = (v_0{:} v_1{:} v_2{:} v_3)$, both given by their homogenous coordinates, are incident if and only if:

$$x_0 v_0 + x_1 v_1 + x_2 v_2 + x_3 v_3 = 0 \quad (4)$$

and the symmetry of the equation (4) reflects the duality in the projective space.

Equation (4) also shows that the ideal plane has the coordinates in the form $(v_0{:}0{:}0{:}0)$ with $v_0 \neq 0$, while all other planes are uniquely determined by their homogenous coordinates; the triple (v_1, v_2, v_3) represents the normal line of the plane and the coordinates of planes through the origin point are characterized by $v_0 = 0$.

2.4. Algebraic geometry of the projective space

From the definition of homogenous coordinates (1) of the points in $\mathbf{P}^3(\mathbb{R})$ we see that the mapping:

$$(x_1, x_2, x_3) \rightarrow (1{:}x_1{:}x_2{:}x_3) \quad (5)$$

is an embedding of three-dimensional vector space $\mathbb{R}^3$ into $\mathbf{P}^3(\mathbb{R})$.

To define the action of a polynomial f at a point of the projective space given by homogenous coordinates as in (1), the following must hold:

$$f(\lambda x_0{:} \lambda x_1{:} \lambda x_2{:} \lambda x_3) = \lambda^4 f(x_0{:}x_1{:}x_2{:}x_3), \lambda \neq 0, \lambda \in \mathbb{R} \quad (6)$$

Polynomials in four variables that satisfy the condition (6) are called *homogenous polynomials* of degree 4, and a point $x = (x_0{:}x_1{:}x_2{:}x_3)$ is a *root of the homogenous polynomial f* if $f(x_0, x_1, x_2, x_3) = 0$.

Sets consisting of roots in $\mathbf{P}^3(\mathbb{R})$ of one or more homogenous polynomials are called *algebraic sets* [26]. Those algebraic sets that can be defined by one equation are called *algebraic surfaces* [25].

2.5. Polarities and null systems in the extended Euclidean space

There are two main types of transformations in the projective space [24]. First, the automorphisms of the space – bijective transformations that preserve the incidence relation; every point is transformed into another point such that collinear points have collinear images. Such transformations are called *projective collineations*. Their action on the set of points determines their action on the sets of lines and planes since a line can be determined by two and a plane by three points and the preservation of

incidence relation guarantees that the image of a line is a line and the image of a plane is a plane.

All projective collineations of the extended Euclidean space $\mathbf{P^3(\mathbb{R})}$ form a group (an algebraic structure on a set with one defined operation; the inverse and the composition of collineations will again be a collineation) and this group is isomorphic to the group of regular 4×4 real matrices, precisely to its quotient by the group of real diagonal 4×4 matrices, since every such diagonal matrix will correspond to the identity collineation. Therefore, the group of projective collineations of $\mathbf{P^3(\mathbb{R})}$ is the projective linear group $PGL_4(\mathbb{R}) = GL_4(\mathbb{R})/\lambda\mathbb{R}^4$.

Duality between points and planes in the projective space brings forth the other type of transformations of projective space and these are the bijective transformations from the set of points to the set of planes which preserve incidence relation. These type of transformations are called *projective correlations*. Line, determined by two points, is transformed by a correlation to the intersection line of the planes that are the images of the two defining points. The inverse transformation of a correlation is again a correlation while the composition of two correlations is a collineation.

We call the points lying in a plane which is the image of some point A *conjugate* to the point A, and those points lying in their own image plane are called *self-conjugate*. Pairs of associated lines are called *conjugate* lines.

Those correlations that are involutions, transformations that composed to themselves give identity map, are called polarities [24, 27, 28].

2.5.1 Analytic approach

We represent a point x = $(x_0:x_1:x_2:x_3)$ by a vector x = $(x_0, x_1, x_2, x_3)^T$ a plane $\upsilon = (\upsilon_0: \upsilon_1: \upsilon_2: \upsilon_3)$ by a vector $\upsilon = (\upsilon_0: \upsilon_1: \upsilon_2: \upsilon_3)^T$ and rewrite the condition of incidence (4) in the matrix form as:

$$x^T\upsilon = 0 \text{ ili } \upsilon^Tx = 0 \tag{7}$$

Let ϱ be a correlation of the extended Euclidean space $\mathbf{P^3(\mathbb{R})}$. The restriction of this transformation to the set of points (which will be denoted as v' to emphasize that the codomain is the set of planes) can be represented by a regular real 4×4 matrix [28]:

$$\varrho\dots\upsilon' = Ax, \ det(A) \neq 0 \tag{8}$$

The same is true for the restriction of $\mathbf{\varrho}$ to the set of planes, we again have a 4×4 regular matrix such that:

$$\varrho'\dots x' = B\upsilon, \ det(B) \neq 0 \tag{9}$$

Since the incidence relation must be preserved, (7), (8) and (9) imply:

$$(Ax)^T B\upsilon = 0 \ /^T$$
$$\tag{10}$$
$$\upsilon^T(B^TA)x = 0.$$

and it follows that B = A^{-T}, if the matrix A represents the restriction of the correlation to the set of points, then A^{-T} will be the matrix corresponding to the restriction of the same correlation to the set of planes.

If ϱ is a polarity, an involuntary correlation, then $\varrho' \circ \varrho = \pi$, where π is the identity collineation, from (7), (8) and (9) we read $\pi\dots x' = A^{-T}Ax$ and the property of homogenity of coordinates implies $A^TA = \lambda Id$, therefore, $\lambda = \pm1$. If $\lambda = 1$, then the matrix A is symmetric, i.e. $A=A^T$, while in the case $\lambda = -1$ it is skew-symmetric, i.e. $A = A^T$. Polarities defined by symmetric matrices can be of three types [28], with respect to the existence of self-conjugate points. Given a point x, all points conjugate to x must satisfy the equation:

$$y^TAx = 0 \tag{11}$$

Self-conjugate points are characterized by:

$$x^TAx = 0 \tag{12}$$

and equation (12) defines a quadric in $\mathbf{P^3(\mathbb{R})}$.

Polarities defined by skew-symmetric matrices are called *null polarities* (null systems). In the case of null polarity, all points are self-conjugate and we have the following terminology; a point is called a *null point* of its corresponding plane, and the plane is called the *null plane* of its corresponding point [29]. If a line is determined by two points, its corresponding line is the intersection line of their null planes. If a line lies in a null plane of any of its points, then the correlation maps it to itself and we call such lines *null lines*.

3. Line space

In subsection 2.3 we defined homogenous coordinates for points and planes in the extended Euclidean space $\mathbf{P^3(\mathbb{R})}$ so we can consider either of them to be "basic" objects. On the other hand, lines are derived objects and can be observed as join of two points or dually as meet of two planes. In terms of projective geometry, a line is a carrier of its range of points, consisting of all points incident with the line or dually a line is a carrier of a pencil (sheaf) of planes, consisting of all planes incident with the line. If we want to have a geometric reinterpretation of the extended Euclidean space $P^3(R)$ such that lines become "basic" elements, then line geometry is the answer to our desire.

Skup svih pravaca proširenoga euklidskog prostora $\mathbf{P^3(\mathbb{R})}$, zajedno s relacijom incidencije, to jest geometrijskom strukturom naslijeđenom iz projektivnoga prostora $\mathbf{P^3(\mathbb{R})}$, nazivamo *pravčastim prostorom*.

The set of all lines of the extended Euclidean space $\mathbf{P^3(\mathbb{R})}$, together with the inherited incidence relation, is called *line space*. In the three-dimensional projective space $\mathbf{P^3(\mathbb{R})}$ both the set of points and the set of lines are three-parameter set, as can be seen from their coordinatizations, and we say that $\mathbf{P^3(\mathbb{R})}$

contains points and planes. But the set of lines is bigger! A pencil of lines through a point i.e. all the lines that are incident with one point, has ∞^2 lines. Since every line lies in ∞ pencils, because it carries a range of points, we count that there are $\infty^3 \cdot \infty^2/\infty^1 = \infty^4$ lines in $\mathbf{P^3(\mathbb{R})}$.

Objects of interest in the extended Euclidean space $\mathbf{P^3(\mathbb{R})}$ are continuously connected sets of points, namely, curves (sets of ∞^1 continuously connected points, i.e. one-parameter sets of points) and surfaces (sets of ∞^2 continously connected points, i.e. two-parameter sets of points). In line space, we are interested in similar types of formations [28, 29]:

- <u>ruled surface:</u> set of ∞^1 continuously connected lines, i.e. one–parameter set
- <u>congruence:</u> set of ∞^2 continuously connected lines, i.e. two-parameter set
- <u>complex:</u> set of ∞^3 continuously connected lines, i.e. three-parameter set.

Ruled surfaces are surfaces in the common meaning, sets of points, and they were studied long before line space came to be. On the other hand, congruences and complexes never had such attention. One reason is the complexity of their structure, given that as sets of points they are equal to the whole space which makes their visualisation extremely complicated.

They appeared as objects of interest in Plücker's paper [3] from 19th century where he first considered lines as basic object in space, following Grassmann's idea on coordinatization of linear r-dimensional subspaces of vector spaces of dimension n, r < n, and by studying linear and quadratic equations in line coordinates he defined line congruences and complexes.

3.1. Plücker coordinates of a line

Choosing a coordinate system for the extended Euclidean space $\mathbf{P^3(\mathbb{R})}$ is equivalent to choosing a basis for the vector space $\mathbb{R}^4$. Let $e_0, .., e_3$ be a basis for $\mathbb{R}^4$. The outer product is an operation which takes two vectors and assign to them a new vector in another vector space which we denote by $\Lambda^2\mathbb{R}^4$, and the resulting vector of the two vectors x, y $\in \mathbb{R}^4$, their outer product, will be denoted by x $\wedge$ y. This operation must satisfy the following two properties for all vectors x, y, z $\in \mathbb{R}$ and all real numbers α, β:

$$\mathbf{x} \wedge (\alpha\mathbf{y}+ \beta\mathbf{z}) = \alpha\mathbf{x} \wedge \mathbf{y}+\mathbf{x} \wedge \beta\mathbf{z} \text{ i } (\alpha\mathbf{x}+ \beta\mathbf{y}) \wedge \mathbf{z} = \wedge \alpha\mathbf{x} \wedge \mathbf{z}+ \beta\mathbf{y} \wedge \mathbf{z} \quad (13)$$

property called *bilinearity*, meaning the operation is linear in both arguments, and

$$x \wedge y = -y \wedge x \quad (14)$$

property called *skew-commutativity* [4].

The two properties (13) and (14) guarantee that the space $\mathbf{\Lambda}^2\mathbb{R}^4$ is six-dimensional ([28], lemma 2.1.1), and the following six vectors make a basis:

$$e_0 \wedge e_1, e_0 \wedge e_2, e_0 \wedge e_3, e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2 \quad (15)$$

If a point in $\mathbf{P^3(\mathbb{R})}$ is represented as an ordered four-tuple of real numbers, or equivalently as a vector x = (x_0, x_1, x_2, x_3), then its basis representation in $\mathbb{R}^4$ is $\mathbf{x} = \sum_{i=0}^{3} x_i e_i$. Basis representation of the outer product of two points x and y if the basis is given as in (15), is as follows:

$$x \wedge y = (x_0y_1-x_1y_0) e_0 \wedge e_1 + (x_0y_2-x_2y_0) e_0 \wedge e_2 + (x_0y_3-x_3y_0)e_0 \wedge e_3$$
$$+ (x_2y_3-x_3y_2) e_2 \wedge e_3 + (x_3y_1-x_1y_3) e_3 \wedge e_1+ (x_1y_2-x_2y_1) e_1 \wedge e_2 \quad (16)$$

Coefficients appearing in (16) can be interpreted as 2×2 sub-determinants of 2×4 matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix} \quad (17)$$

If coefficients in (16) are denoted by l_{ij}, meaning they correspond to the basis vector $e_i \wedge e_j$, then an element of the form

$$L = (l_{01}, l_{02}, l_{03}, l_{23}, l_{31}, l_{12}) \in \Lambda^2\mathbb{R}^4 \quad (18)$$

will correspond to a line in $\mathbf{P^3(\mathbb{R})}$ if and only if ([28], lemma 2.1.2):

$$l_{01}l_{23} + l_{02}l_{31} + l_{03}l_{12} = 0 \quad (19)$$

The relation (19) is called *Plücker identity*.

If a line is defined as a join of two points, then the coefficients of the vector x $\wedge$ y in (16) are called *Plücker line coordinates*. Should we choose a different pair of points in the span determined by the line, then the new coordinates will differ from the previous ones by the same scalar factor ([28], lemma 2.1.2.). Hence, we denote the Plücker line coordinates in the homogenous form

$$(l_{01} : l_{02} : l_{03} : l_{23} : l_{31} : l_{12}) = (l,\bar{l}) \quad (20)$$

With $\boldsymbol{l}$ = (l_{01}, l_{02}, l_{03}) and $\overline{\boldsymbol{l}}$ = (l_{23}, l_{31}, l_{12}).

If a line L is given as a meet of two planes with homogenous coordinates as in (3), or in vector form as u = (u_0, u_1, u_2, u_3) and v = (v_0, v_1, v_2, v_3) in $\mathbb{R}^4$, then, like in (16), we can define their outer product u $\wedge$ v, and this vector is denoted by

$$(l^*_{01}, l^*_{02}, l^*_{03}, l^*_{23}, l^*_{31}, l^*_{12}) = u \wedge v \quad (21)$$

Dual Plücker line coordinates of the line L (axial coordinates) are given by $(l^*_{01}, l^*_{02}, l^*_{03}, l^*_{23}, l^*_{31}, l^*_{12})$ and the following identity holds ([28], lemma 2.1.4):

$$(l^*_{01}, l^*_{02}, l^*_{03}, l^*_{23}, l^*_{31}, l^*_{12}) = (l^*_{23}, l^*_{31}, l^*_{12}, l^*_{01}, l^*_{02}, l^*_{03}) = (\overline{\boldsymbol{l}}, \boldsymbol{l}) \quad (22)$$

With $(\boldsymbol{l}, \overline{\boldsymbol{l}})$ being Plücker line coordinates of the line L as in (20).

3.2. Klein model

In the former subsection 3.1 we attached to each line in $\mathbf{P^3(\mathbb{R})}$ an ordered homogenous six-tuple of numbers (20), but from all possible six-tuples, only those that satisfy (19) correspond to lines of $\mathbf{P^3(\mathbb{R})}$. That way, a mapping can be defined from the set of lines to the five-dimensional projective space $\mathbf{P^5(\mathbb{R})}$ such that a line is mapped to a point of $\mathbf{P^5(\mathbb{R})}$ defined by line's homogenous coordinates. First person who was aware of this map was Felix Klein, hence the mapping has the name *the Klein mapping*.

The relation (19) can be viewed as a quadratic equation in Plücker coordinates and it defines a quadric in $\mathbf{P^5(\mathbb{R})}$ which is called *the Klein quadric* and we denote it by M_2^4. The following theorem holds ([28], Theorem 2.1.6):

Set of lines of the extended Euclidean space $\mathbf{P^3(\mathbb{R})}$ is in bijective correspondence, via the Klein mapping, to the set of points in the Klein quadric $M_2^4 \subset \mathbf{P^5(\mathbb{R})}$.

Thus we have a point model of line space.

3.3. Linear complexes and null systems

In subsection 2.5.1. we defined a null polarity as a polarity where all points are self-conjugate and we introduced the terminology of null points, null lines and null planes. Using analytical method, null polarity is defined by a skew-symmetric 4x4 real matrix. A skew-symmetric matrix $A = (a_{ij})_{0\le i,\,j\le 3}$ is such that $_{ii} = 0$ and $a_{ij} = -a_{ji}$ hence it is defined by six parameters $a_{01}, a_{02}, a_{03}, a_{23}, a_{31}, a_{12}$. We introduce the following notation:

$$\bar{\mathbf{a}} = (a_{01}, a_{02}, a_{03}) \text{ i } \mathbf{a} = (a_{23}, a_{31}, a_{12}) \qquad (23)$$

A null line is a self-conjugate line, an invariant line. Such lines lie in the null plane of one of its points. A line is a null line if and only if the line is a join of two conjugate points, since it must be the intersection line of their null planes and these two planes must contain the original two points.

Every point in space carries a pencil of null lines consisting of all lines incident with the point's null plane. We see that there is ∞^3 null lines in space and they form a linear complex [29]. We will prove the existence of this complex via the analytical method.

We can characterize null lines using their homogenous coordinates $L = (\mathbf{l}, \overline{\mathbf{l}})$ defined in (20):

If a null system is given by a skew-symmetric matrix $A = (a_{ij})_{0\le i,\,j\le 3}$, then a line L is a null line if and only if

$$\bar{\mathbf{a}} \cdot \mathbf{l} + \mathbf{a}\cdot \overline{\mathbf{l}} = 0 \qquad (24)$$

The proof of the former claim is strictly computational; we look at a null line as a join of two conjugate points and the claim follows from the condition for point conjugacy (13) and the definition of line's homogenous coordinates as outer product of the two points (16) ([28], lemma 3.1.2.).

The image L' = (l', $\bar{\mathsf{l}}$') of the line L by a null system defined with a skew-symmetric matrix $A = (a_{ij})_{0\le i,\,j\le 3}$ is given by

$$(\mathbf{l'}, \overline{\mathbf{l}}') = (\mathbf{a} \cdot \bar{\mathbf{a}}) \,(\mathbf{l}, \overline{\mathbf{l}}) - (\bar{\mathbf{a}} \cdot \mathbf{l} + \mathbf{a} \cdot \overline{\mathbf{l}})(\mathbf{a}, \bar{\mathbf{a}}) \qquad (25)$$

and the associated pair of lines L and L' is called *reciprocal pair*.

The equation (24) is a linear equation in Plücker line coordinates and the set of lines L that satisfy the equation is called a *linear line complex.* The vector (a,ā) is called the *homogenous coordinate vector* of the complex L.

If the vector (a, ā)) corresponds to a line in $\mathbf{P^3(\mathbb{R})}$, which means it satisfies the Plücker identity (19), then we say that the linear complex L is a *singular linear complex* and it consists of all lines in $\mathbf{P^3(\mathbb{R})}$ that intersect the line (a, ā), which is then called *axis of the singular linear complex.* The matrix $A = (a_{ij})_{0\le i,\,j\le 3}$ obtained as in (23) is in this case singular and it cannot define a null polarity because of (19).

If the vector (a, ā), defines a null polarity, then the linear complex $\mathcal{L}$ is called a *regular linear complex.*

3.3.1. Geometry of the null system

All lines belonging to a regular linear complex $\mathcal{L}$ are null lines of the attached null polarity. All remaining lines $\mathcal{L}$ in space, not belonging to the complex $\mathcal{L}$, have a reciprocal line L' such that the null plane of every point incident with L contains the line L' and vice versa, all planes incident to the line L have null points on the line L'.

If a line in space has an ideal line as the reciprocal line, then we call it a *diameter* of the null polarity. Planes incident with a diameter have ideal points as their null points and such planes are called *diameter planes*. All diameters pass through the null point of the ideal plane, therefore, there are ∞^2 diameters in every null polarity and they are all mutually parallel [29].

Let (a, ā) be the homogenous linear vector of a null polarity. If $L = (\mathbf{l}, \overline{\mathbf{l}})$ is a diameter of that null polarity, then we must have l=a, since (23) and $\mathbf{l'} = (0, 0, 0)$ imply

$$\mathbf{l} = \frac{\bar{\mathbf{a}} \cdot \mathbf{l} + \mathbf{a} \cdot \overline{\mathbf{l}}}{\mathbf{a} \cdot \bar{\mathbf{a}}}\, \mathbf{a} \qquad (26)$$

Null plane of every point lying on a diameter must contain its ideal reciprocal line, therefore, we can attach to each diameter a pencil of mutually parallel planes. There exists one diameter orthogonal to its attached pencil of parallel planes and we call that diameter *axis of null polarity*.

The following claim is true ([28], theorems 3.1.6. and 3.1.9.):
Plücker coordinates of the axis of null polarity are given by

$$(\mathbf{a}, \bar{\mathbf{a}} - (\mathbf{a}\cdot \bar{\mathbf{a}}/a^2)\mathbf{a}) \qquad (27)$$

where (a, ā) *is the homogenous linear vector of that null polarity.*

4. Applications in statics

Geometric interpretation of Plücker coordinates of lines in $\mathbf{P^3(\mathbb{R})}$ can be grasped through the standard metric of the Euclidean space $\mathbf{E^3}$ and algebraic structure of the vector space $\mathbb{R}^3$.

Ordered triplets $\boldsymbol{l} = (l_{01}, l_{02}, l_{03})$ and $\overline{\boldsymbol{l}} = (l_{23}, l_{31}, l_{12})$ corresponding to Plücker coordinates of line $L = (\boldsymbol{l}, \overline{\boldsymbol{l}})$ can be interpreted as vectors in $\mathbb{R}^3$ and then the identity (19), which can be rewritten in the vector form as $\boldsymbol{l} \cdot \overline{\boldsymbol{l}} = 0$, is precisely the condition of orthogonality of these two vectors.

Ideal point of the line $L = (\boldsymbol{l}, \overline{\boldsymbol{l}})$ is the intersection point of that line and the ideal plane. Using Plücker line coordinates and homogenous coordinates of the ideal plane (1:0:0:0) we can compute that the homogenous coordinates of line's ideal point are $(0{:}l_{01}{:}l_{02}{:}l_{03})$. In the context of the vector space R^3 we see that the direction of the line L is determined with the vector l, since line's ideal point is the intersection point of that line with every line parallel to it.

If a line is given as a join of one (non-ideal) point $x = (1{:}x_1{:}x_2{:}x_3)$ and its ideal point $(0{:}l_{01}{:}l_{02}{:}l_{03})$, we get the following vector equality in $\mathbb{R}^3$:

$$\overline{\boldsymbol{l}} = x \cdot \boldsymbol{l} \tag{28}$$

where $x = (x_1{:}x_2{:}x_3) \in \mathbb{R}^3$ is the vector of affine coordinates of the point x. Hence we have the following:

If $\boldsymbol{S}$ is a force acting along line $\|L\|$, with intensity and placed at the point x, then the vector $\overline{\boldsymbol{l}}$ is the moment vector of this force with respect to the origin point.

In other words, a force can be represented as

$$\mathbf{S} = (\mathbf{s}, \mathbf{\bar{s}}) \tag{29}$$

Where s is the force vector and s the moment vector relative to the origin. The representation (29) is called *force coordinates* of the force **S.** Force coordinates are not homogenous since the force is, beside the line of action, determined by its intensity and orientation, and on each line in space we have ∞^1 forces. In total, there are ∞^5 forces in space.

4.1. Statical equivalence and linear complexes

If a system of forces $(s_i, \bar{s}_i)$ acting in space is given, then we say it is equivalent to another system $(g_j, \bar{g}_j)$ if:

$$\sum s_i = \sum g_j \ \text{i} \ \sum \bar{s}_i = \sum \bar{g}_j \tag{30}$$

Let $(s_i, \bar{s}_i)$, be a system of forces and let . We call $(r, \bar{r}) = (\sum s_i, \sum \bar{s}_i)$ the *principal system* (*relative to the origin*).

We say that a system of forces is *generic* if:

$$r \neq (0, 0, 0) \ \text{i} \ r \cdot \bar{r} \neq 0 \tag{31}$$

A generic system of forces cannot be reduced to a single force (resultant force), since the second condition says that $(r, \bar{r})$ is

not a line, because it fails the Plücker identity (19), therefore, the principal system (relative to the origin) is composed of a principal force and a principal moment (relative to the origin). If the second condition is not satisfied, $r \cdot \bar{r} = 0$, or, geometrically, if the force vector and the moment vector are mutually orthogonal, then the system can be reduced to one resultant force. If the system doesn't satisfy neither of the two conditions, then we have two possibilities: if $\bar{r} = (0, 0, 0)$ then the system is in equilibrium, and in the second case if $\bar{r} \neq (0, 0, 0)$ the system can be reduced to a resultant couple with moment vector $r\bar{r}$.

The moment of the force $S = (s, \bar{s})$ relative to a point p given by affine coordinates $p = (p_1, p_2, p_3)$ is computed as

$$\bar{s}_p = \bar{s} - p \times s \tag{32}$$

Basic principles of statics (see for instance [30]) tell us that computation of the principal moment of a system of forces $(s_i, \bar{s}_i)$ relative to a point p can also be done using (32), as long as we replace $(s, \bar{s})$ with $(r, \bar{r})$.

Principal moments relative to different points can only differ by a component orthogonal to the line of action of the principal force, as we see from (32). Vector product of parallel lines vanishes, therefore, principal moments relative to points lying on a line parallel to the line of action of the principal force will be the same but will differ for different parallel lines.

Components of the principal moments parallel to the principal force are equal, therefore, principal moments parallel to the principal force must have the lowest intensity and these are the moments lacking a component orthogonal to the principal force. Also, the points relative to which these moments are taken must lie on a line parallel to the principal force (and also to the moments) and this line is called the *central axis* [30]. Unlike the force vectors which are "line bound", moment vectors are free vectors, nevertheless, we visualize them as acting at the points relative to which they are taken. Thus visualized moment vector field is "governed by" a rotational symmetry about the central axis.

Principal moment relative to a point **p** lying in a plane orthogonal to the central axis (intersection point **o** of this plane and the central axis excluded) can be resolved into one component lying in the plane and another component orthogonal to the plane. The component in the plane is orthogonal to the central axis and also to the joining line of the intersection point **o** and the point **p**. As we mentioned, the components orthogonal to the plane are all equal. On the other hand, intensities of components lying in the plane increase proportionally as we go further from the intersection point, but are equal on a circle with centre **o**. Principal moments with respect to the points on this circle have the same slope (Figure 3.), which depends on the intensity of the principal force and principal moment (relative to the point **o**) and on the radius of the circle – the greater the radius, the smaller the slope. If we visualize these moments, we see that they lie on one system of rulings of a rotational hyperboloid whose central axis is the central axis of the system of forces while the smallest circle of latitude of the hyperboloid is the

circle defining the slope (Figure 4.) and plane components of moment vectors lie on tangents to the circle.
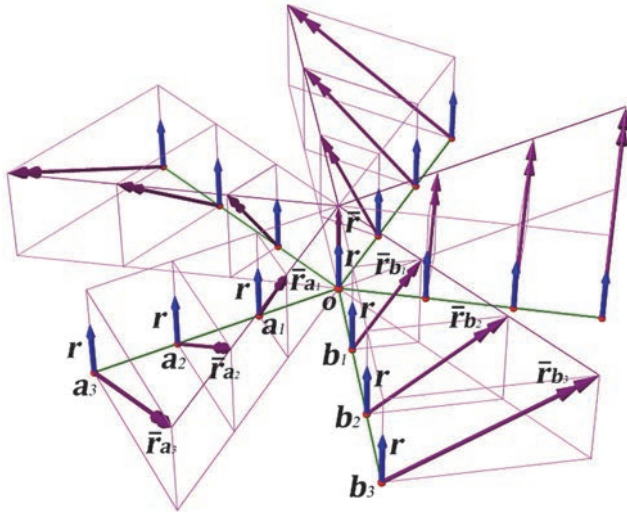

Figure 3. Principal systems with respect to points lying in a plane orthogonal to the central axis
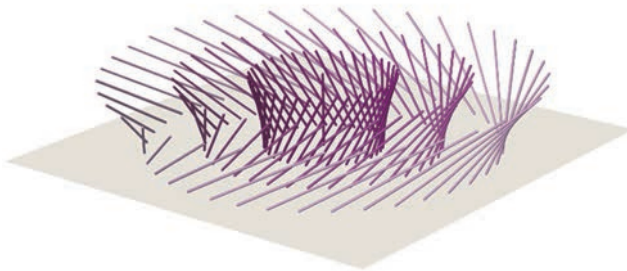

Figure 4. Congruence of lines that carry the principal moments relative to the points in a plane if the moments are positioned in that points

When the plane is translated in the direction of the central axis, the figure remains the same (Figure 5). The lines carrying the principal moments relative to the points of the plane (if we place them at corresponding points) will form a line congruence which will contain all rulings from one system of rulings of a series of hyperboloids having the same central axis and concentric smallest circles of latitude (Figure 4), while the lines carrying principal moments with respect to all the points in space will form a line complex. (Program functions used for creation of figures 3, 4 and 5 are written in *SageMath* software [31]).

This line complex in not the regular linear complex R attached to the null polarity defined by the homogenous linear vector (r, r̄) in the sense of subsection 3.3 (it is not even linear).

Magnitude of the moment of the force (s, s̄) about an oriented line (c, c̄) is defined by

$$m\big((s,\bar{s})(\vec{c},\overline{c})\big)=\frac{1}{\|c\|}\big(c\cdot\bar{s}+\overline{c}\cdot s\big) \tag{33}$$

while the magnitude of the moment of the system of forces (s_i, $\bar{s}_i$) can be computed by the same expression if we replace (s, s̄) by the principal system (**r, r̄**).

Condition for vanishing of the moment of the system of forces with the principal system (r, r̄) about an oriented line $(\vec{c},\overline{c})$ is

$$c\cdot\bar{r}+\overline{c}\cdot r=0 \tag{34}$$

If we replace (**r, r̄**) by (**a, ā**) and (**c, c̄**) by (**l, l̄**) in the former expression, we see it is precisely the expression (24) used to characterize null lines of a null polarity, i.e. the lines of a regular linear complex. Therefore, the complex R consists of all lines in space, with given orientation, such that the moment of the system of forces (s_i, $\bar{s}_i$) vanishes, and this was the motivation for Möbius to give the term null lines to such lines in his paper [1]. The axis of the linear complex given by (27) is the central axis of the system of forces.
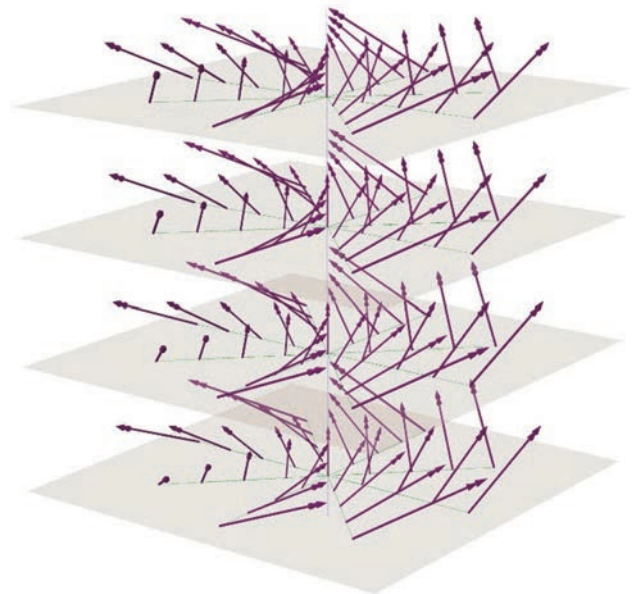

Figure 5. Principal moments relative to points of parallel planes orthogonal to the central axis

To visualize the complex $\mathcal{R}$, in the "point of application" of each principal moment (Figures 3, 4 and 5) we visualize a plane orthogonal to the moment. This plane is the null plane of the "point of application" and the "point of application" is the null point of the plane. Pencil of lines through the "point of application" lying in that plane is a pencil of null lines; these are the only null lines incident either to the point or to the plane. Null lines must lie in this plane and the reason is the following: principal moments can be replaced by a force couple in this plane; the moment of the couple about a line vanishes only if both forces of the couple intersect the line. Since the principle system contains both the principal moment and the principal force, moment about any line will vanish only if the principal force intersects this line.

The following claim is also true ([28], Proposition 3.4.8.):

A generic system of forces with principal system (r,r) *is statically equivalent to a system of two forces, such that the forces act along a pair of reciprocal lines of the regular linear complex* R.

The proof goes as follows.

There are infinitely many lines in space not belonging to the complex R defined by the homogenous linear vector (r,r). Let L be a line not belonging to R such that its conjugate line in the attached null polarity is not ideal. Let L = $(l, \bar{l})$ denote Plücker coordinates of this line. From (24) we see that $\theta = \bar{m} \cdot l + m \cdot \bar{l} \neq 0$. If we put $\varphi = r \cdot r$ then $\varphi \neq 0$. By applying (25) we can find the Plücker coordinates of the conjugate line to be $(l', \bar{l}) = \varphi(l, \bar{l}) - \theta(r, \bar{r})$.

Finally, the equivalence of the given system and the system of two forces lying on the reciprocal pair L and L′ follows from the equation:

$$\left(r, \bar{r} \right) = \frac{\varphi}{\theta} \left(l, \bar{l} \right) - \frac{1}{\theta} \left(l', \bar{l'} \right) \tag{35}$$

4.2. Applications of spatial constructive procedures

In the paper [32] we have described graphical procedures, based on geometrical constructions, for replacing a system of forces with an equivalent one, which can be regarded as partial three-dimensional extensions of the construction of funicular polygon. Geometric constructions used are based on the following two principles:

Principle 1: *Single force can be resolved into two components along two given lines if and only if its line of action and the two given lines are concurrent and coplanar* (Figure 6.). In this case, the system consisting of one force defines a singular linear complex that consists of all lines intersecting the force's line of action.
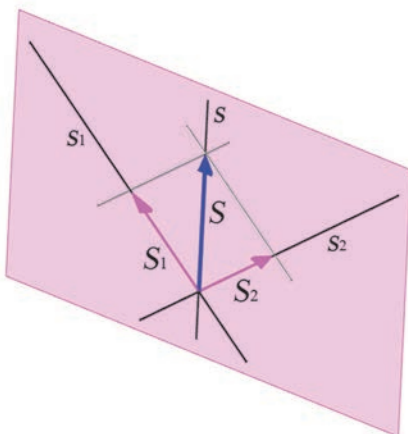

Figure 6. Resolving a force S into two components S$_1$ and S$_2$ using parallelogram rule

Principle 2: *When constructing funicular polygon, each of two given forces is resolved into two components in such a way that one component of the first force and one component of the second force*

cancel each other *(these two components lie on the same line and are equal in magnitude, but opposite in sense)* (Figure 7.).
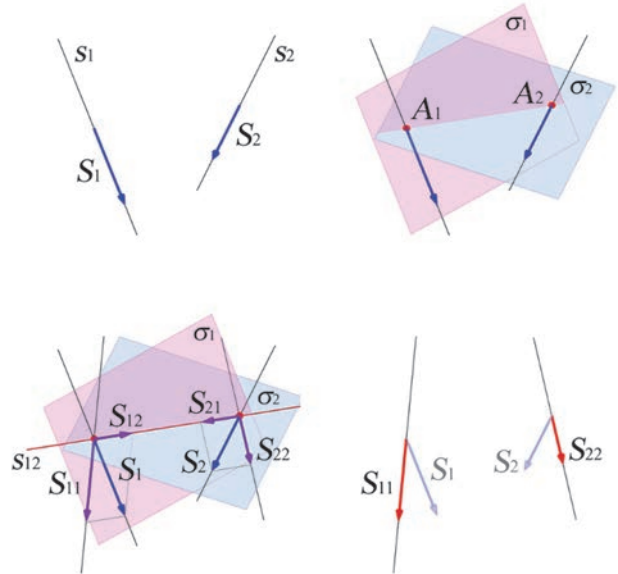

Figure 7. Replacing a system of two forces

By the principle 1, the force and the resolving components lie in the same plane. On lines s_1 and s_2 lines of action of forces S_1 and S_2, we choose two points A_1 and A_2, one on each line. The force S_1 is resolved by principle 1 in the point A_1 into components S_{11} and S_{12} lying in the plane σ_1, determined by the point A_2 and the line s_1, whereas the force S_2 is resolved in the point A_2 into components S_{21} and S_{22} lying in the plane σ_2, determined by the point A_1 and the line s_2, in such way that the components S_{12} and S_{21}, that must lie on the intersecting line of the two planes, cancel each other.

4.2.1. Generic system of three forces with skew lines of action

A generic system of three forces lying on three skew lines will be replaced by a system of two forces lying on reciprocal lines. Geometric construction is as follows (Figure 8.):

Three given forces S_1, S_2, S_3 act along lines s_1, s_2, s_3. By the formerly described procedure we replace forces S_1 and S_2 with force S_{11} and force S_{22}, which will be denoted by S_{23} for the sake of clarity.

The intersection point of the third line s_3 and the plane σ_2 will be denoted by A_3 and we resolve the force S_3 acting at the point A_3 in the plane σ_3, determined by the line s_3 and the point A_2, by the first principle into components S_{32} and S_{33} in such way that, by the second principle, the components S_{23} and S_{32}, lying on the intersection line of the two plane, cancel each other. We have replaced the system S_1, S_2, S_3 with an equivalent system of two forces $S_{11} = R_1$ and $S_{33} = R_2$ that act along a pair of reciprocal lines of the null system defined by the given system of forces.

When a generic system consists of more than three forces, the "classical" procedure of constructing a planar funicular polygon cannot be used since neither one of lines of action of forces S_{11} or S_{33} will intersect the line of action of the fourth force. However, we can apply the previously described procedure to the system consisting of this fourth force and the two forces S_{11} and S_{33}. Thus, every generic system of forces in space can be replaced by two forces such that their lines of action are reciprocal lines.

4.2.2. Generic system of two forces

Given any generic system of two forces, conditions (31) imply that forces' lines of action must be skew and make a reciprocal pair in the attached null polarity.

Such system can be replaced by an equivalent system such that certain elements of the system are fixed, given in advance. So far we have shown how a system of two forces can be replaced with another system of two forces such that one of the forces passes through a given point and the other lies in a given plane (the point mustn't lie in the plane) and discussed the two special cases (case of an ideal point or ideal plane) or such that one force's line of action is given [32].

If one line of action is given, we know that the other force acts along the reciprocal line, therefore, this particular procedure can be used as the geometrical procedure of finding the reciprocal line of a given line when a null polarity is given. Since this construction is on point regarding this paper, we will repeat it here (Figure 9.):

Let s_0 be the given line and S_1, S_2 given forces acting along lines s_1, s_2. We choose two points A_1 and A_2, one on line s_1 and the other on s_2 and denote the line joining them by s_{12}. We resolve S_1 at the point A_1 by principle 1 into component S_{11} lying on the intersecting line s_{01} of the plane σ_0, determined by the point A_1 and a line s_0, and the plane σ_1, determined by lines s_1 and s_{12}, and the component S_{12} on the line s_{12}. By the same principle, we resolve the force S_2 at the point A_2 into components S_{21} and S_{22} in such a way that S_{12} and S_{21} cancel each other by principle 2, which determines the component S_{22}. Thus, we have replaced the system S_1, S_2 with the system S_{11}, S_{22}.

We denote by A_0 the intersection point of the given line s_0 with the line s_{01}. The
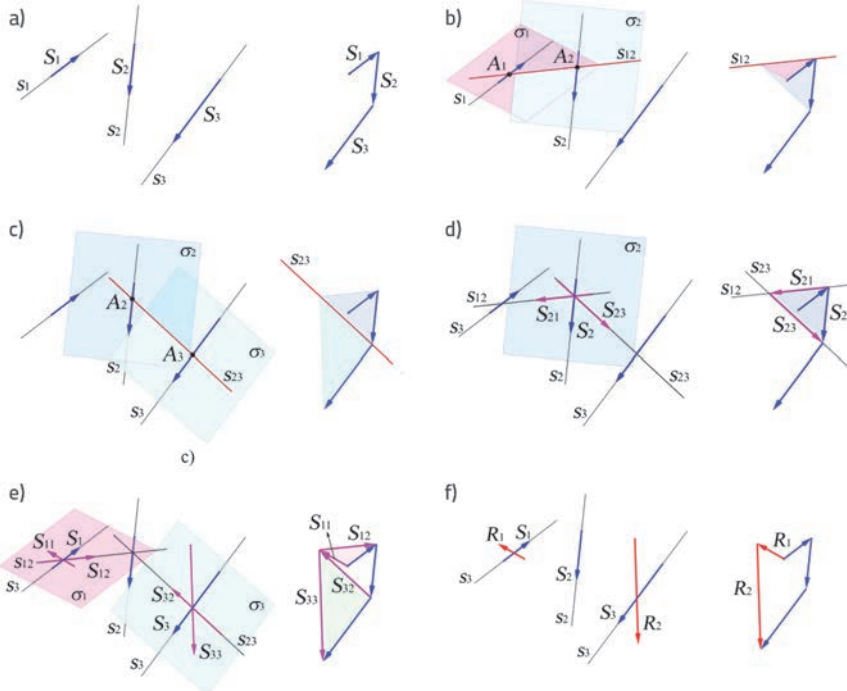

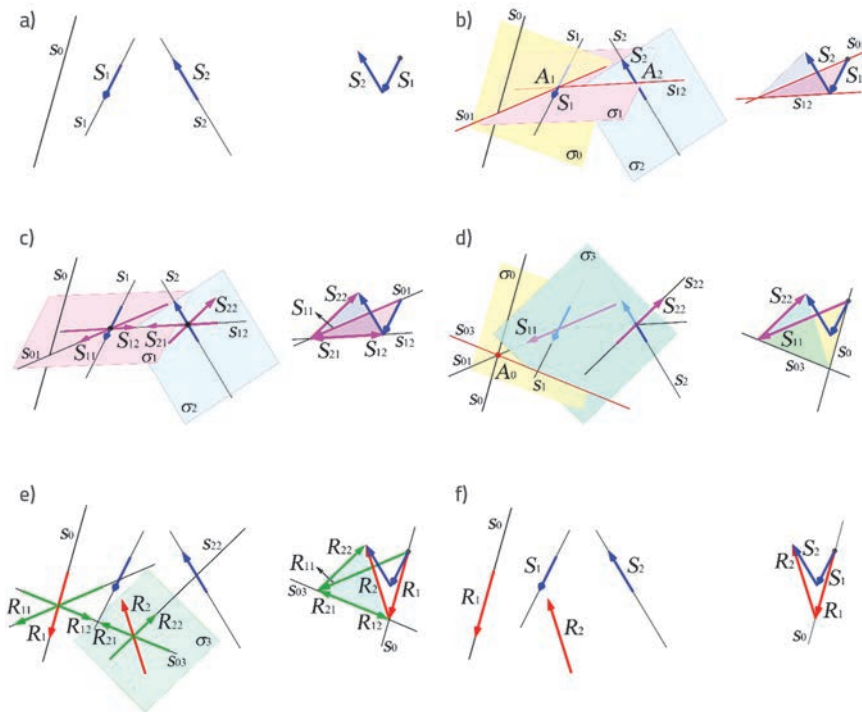**Figure 8. Graphical procedure for replacing three forces with two forces lying on conjugate lines**

Figure 9. Replacing two forces with a force R1 lying on a given line and its reciprocal force R2 [32]

lines s_0, s_{01} and the line s_{03} the intersecting line of the planes σ_0, determined by lines s_0 and s_{01}, and σ_3, determined by the points A_0 and s_{22}, line of action of the force S_{22}, are coplanar (all lie in the plane σ_0) therefore, we can define the force R_1 lying on the line s_0 by its two components R_{11}, lying on the line s_{01} and equal to S_{11}, and R_{12}, acting on a line s_{03}.

The force R_2 is determined by application of parallelogram rule from forces R_{22} equal to force S_{22}, and component R_{21}. Line s_{03} is the intersecting line of planes σ_0 and σ_3 that contain forces R_1 and R_2 as well as their resolving components, hence, by principle 2, R_{21} and R_{12} are equal in magnitude, but opposite in sense.

Two forces R_1 and R_2 which determine a system equivalent to the system S_1, S_2 are lying on a pair of reciprocal lines, i.e. the line of action of R_2 is conjugate to the given line s_0.

As we have mentioned earlier, by means of Grassmann algebra we can directly translate geometrical construction procedures into algebraic expressions and finally into program code. All examples in Figures 7, 8 and 9 are produced using a computer program developed with computer graphic tools *Rhinoceros3D* [33], its plug-in *Grasshopper*, a visual programming tool [34], and *GHPhyton*, a *Grasshopper* plug-in and *Python* interpreter [35].

We can use numerical input data from these examples and produce analytical description of the null polarity, in a way presented in subsection 3.3 and 4.1, to numerically verify that lines of action of resulting forces are conjugate, i.e. are acting along a reciprocal pair of lines.

5. Conclusion

In this paper we deal with line geometry using a combination of synthetic, analytic and algebraic methods. We defined the notion of projective space and work in its model, the extended Euclidean space, where we introduced homogenous coordinates for both points and planes, following the duality principle of the projective space. We analytically defined projective transformation of the space, projective collineations and projective correlations.

We have then defined the line space as the set of lines of the extended Euclidean space, introduced coordinates on this set following Grassmann's idea of outer product and showed that the line space has a point model in five-dimensional projective space. We analytically studied linear complexes and their connection to null polarities, correlations of the extended Euclidean space that are involutions.

In the final section we showed the connection of generic systems of forces (systems that cannot be reduced to a single force) and regular linear complexes, i.e. null polarities. We presented how tools of line geometry can be utilized in solving static problems of replacing systems of forces with equivalent simpler systems.

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